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EXPANSION OF MAGNETOSTATIC POTENTIAL ON MEDIAN PLANE

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Let $R(s)$ be the radial position of the closed orbit assumed to lie in the plane $y=0$. Then, if the radial coordinate $r=R(s)+x$, and the local curvature of $R(s)$ is $k(s) \equiv 1/\rho(s)$, Laplace's equation for the potential becomes¹

$$\frac{1}{1+kx} \cdot \frac{\partial}{\partial x} \left[(1+kx) \frac{\partial \Phi}{\partial x} \right] + \frac{\partial^2 \Phi}{\partial y^2} + \frac{1}{1+kx} \cdot \frac{\partial}{\partial s} \left[\frac{1}{1+kx} \cdot \frac{\partial \Phi}{\partial s} \right] = 0 . \quad (1)$$

Assume an expansion of the form:

$$\Phi = \sum_{m=0} \sum_{n=0} \frac{A_{m,n}(s)}{m!n!} x^m y^n . \quad (2)$$

Substitute this potential into Eq. (1), multiply by $(1+kx)^3$, and adjust the general indices to yield $x^m y^n$ as a factor common to all terms. This yields the following recurrence relation:

$$\begin{aligned} & A_{m+2,n} + (3m+1)kA_{m+1,n} + m(3m-1)k^2A_{m,n} + m(m-1)^2k^3A_{m-1,n} \\ & + A_{m,n+2} + 3mkA_{m-1,n+2} + 3m(m-1)k^2A_{m-2,n+2} \\ & + m(m-1)(m-2)k^3A_{m-3,n+2} - mk'A'_{m-1,n} + A''_{m,n} + mkA''_{m-1,n} = 0 , \end{aligned} \quad (3)$$

where the prime designates differentiation with respect to the longitudinal coordinate s . It is to be noted that all coefficients for which the indices m and n are less than zero are zero.

SPECIFIC LOW ORDER TERMS

The lowest order coefficients $A_{m,0}$ and $A_{m,1}$ are completely arbitrary. These terms in fact represent the fields on the reference curve $^2(x=y=0)$.

$$\begin{aligned}
 A_{00} &= \int B_s ds & A_{01} &= \frac{B_y}{y} \\
 A_{10} &= \frac{B_x}{x} & A_{11} &= \frac{\partial B_y}{\partial x} \\
 A_{20} &= \frac{\partial B_x}{\partial x} & A_{21} &= \frac{\partial^2 B_y}{\partial x^2} \\
 A_{30} &= \frac{\partial^2 B_x}{\partial x^2} & - & - \\
 - & - & - & - \quad (4)
 \end{aligned}$$

Equation (3) serves to relate all other coefficients to those in Eq. (4). Note that n only appears as a subscript.

$n=2$

$$(m=0) A_{02} = -A_{20} - kA_{10} - A''_{00} \quad (5)$$

$$(m=1) A_{12} = -A_{30} - kA_{20} + k^2 A_{10} + k'A'_{00} - A''_{10} + 2kA''_{00} \quad (6)$$

$$\begin{aligned}
 (m=2) A_{22} &= -A_{40} - kA_{30} + 2k^2 A_{20} - 2k^3 A_{10} - 6kk'A'_{00} \\
 &\quad + 4kA''_{10} - 6k^2 A''_{00} + 2k'A'_{10} - A''_{20} \quad (7)
 \end{aligned}$$

$$\begin{aligned}
 (m=3) A_{32} &= -A_{50} - kA_{40} + 3k^2 A_{30} - 6k^3 A_{20} + 6k^4 A_{10} \\
 &\quad + 24k^3 A''_{00} + 36k^2 k'A'_{00} - 18k^2 A''_{10} - 18kk'A'_{10} \\
 &\quad + 6kA''_{20} + 3k'A'_{20} - A''_{30} \quad (8)
 \end{aligned}$$

$$\begin{aligned}
 (m=4) A_{42} &= -A_{60} - kA_{50} + 4k^2 A_{40} - 12k^3 A_{30} + 24k^4 A_{20} - 24k^5 A_{10} \\
 &\quad - 120k^4 A''_{00} - 240k^3 k'A'_{00} + 96k^3 A''_{10} + 144k^2 k'A'_{10} \\
 &\quad - 36k^2 A''_{20} - 18kk'A'_{20} + 8kA''_{30} + 4k'A'_{30} - A''_{40} \quad (9)
 \end{aligned}$$

n=3

$$(m=0) A_{03} = -A_{21} - k A_{11} - A''_{01} \quad (10)$$

$$(m=1) A_{13} = -A_{31} - k A_{21} + k^2 A_{11} + k' A'_{01} - A''_{11} + 2k A''_{01} \quad (11)$$

$$\begin{aligned} (m=2) A_{23} = & -A_{41} - k A_{31} + 2k^2 A_{21} - 2k^3 A_{11} - 6 k k' A'_{01} \\ & + 4k A''_{11} - 6 k^2 A''_{01} + 2k' A'_{11} - A''_{21} \end{aligned} \quad (12)$$

$$\begin{aligned} (m=3) A_{33} = & -A_{51} - k A_{41} + 3k^2 A_{31} - 6k^3 A_{21} + 6k^4 A_{11} \\ & + 24k^3 A''_{01} + 36k^2 k' A'_{01} - 18k^2 A''_{11} - 18k k' A'_{11} \\ & + 6k A''_{21} + 3k' A'_{21} - A''_{31} \end{aligned} \quad (13)$$

$$\begin{aligned} (m=4) A_{43} = & -A_{61} - k A_{51} + 4k^2 A_{41} - 12k^3 A_{31} + 24k^4 A_{21} - 24k^5 A_{11} \\ & - 120 k^4 A''_{01} - 240k^3 k' A'_{01} + 96k^3 A''_{11} + 144k^2 k' A'_{11} \\ & - 36 k^2 A''_{21} - 18k k' A'_{21} + 8k A''_{31} + 4k' A'_{31} - A''_{41} \end{aligned} \quad (14)$$

n=4

$$\begin{aligned} (m=0) A_{04} = & A_{40} + 2k A_{30} - k^2 A_{20} + k^3 A_{10} + 5k k' A'_{00} \\ & - 2k A''_{10} + 4k^2 A''_{00} + 2A''_{20} + k'' A_{10} + A'''_{00} \end{aligned} \quad (15)$$

n=5

$$\begin{aligned} (m=0) A_{05} = & A_{41} + 2k A_{31} - k^2 A_{21} + k^3 A_{11} + 5k k' A'_{01} \\ & - 2k A''_{11} + 4k^2 A''_{01} + 2A''_{21} + k'' A_{11} + A'''_{01} \end{aligned} \quad (16)$$

VECTOR POTENTIAL

In hamiltonian analysis, it is necessary to have an expression for the vector potential. Using L operator analysis³, since i_y is the only constant vector in the present coordinate system, let

$$L = i_y x \nabla \quad (17)$$

with this operator a general decomposition of any vector such as the vector potential may be written as

$$A = LU + \nabla x LV + \nabla W. \quad (18)$$

The flux density becomes

$$B = \nabla x A = \nabla x LU - L \nabla^2 V \quad (19)$$

and

$$\nabla x B = -L \nabla^2 U - \nabla x L \nabla^2 V = 0, \quad (20)$$

which may be satisfied rather generally with

$$\nabla^2 U = 0 \quad (21)$$

$$\nabla^2 V = 0 \quad (22)$$

Then

$$B = \nabla x LU = i_y \nabla^2 U - \nabla \frac{\partial U}{\partial y} = -\nabla \frac{\partial U}{\partial y} = \nabla \Phi \quad (23)$$

Hence

$$U = - \int_0^y \Phi dy' + F(x, s) = - \sum \frac{A_{mn}}{m! (n+1)!} x^m y^{n+1} + F(x, s). \quad (24)$$

To obtain the function, $F(x, s)$ notice⁴ that

$$\nabla^2 U = - \nabla_{x, s}^2 \int_0^y \Phi dy' - \frac{\partial^2}{\partial y^2} \int_0^y \Phi dy' + \nabla^2 F = 0 \quad (25)$$

or, since $\nabla^2 \Phi = 0$,

$$\nabla^2 F = - \int_0^y \frac{\partial^2 \Phi}{\partial y^2} dy' + \frac{\partial \Phi}{\partial y} = -\frac{\partial \Phi}{\partial y} + \left(\frac{\partial \Phi}{\partial y} \right)_{y=0} + \frac{\partial \Phi}{\partial y} = B_y(x, o, s) \quad (26)$$

Thus

$$\frac{1}{1+kx} \frac{\partial}{\partial x} \left[(1+kx) \frac{\partial F}{\partial x} \right] + \frac{1}{1+kx} \frac{\partial}{\partial s} \left(\frac{1}{1+kx} \frac{\partial F}{\partial s} \right) = \sum \frac{A_{ml}(s)}{m!} x^m$$

or

$$\frac{\partial}{\partial x} \left[(1+kx) \frac{\partial F}{\partial x} \right] + \frac{\partial}{\partial s} \left[\frac{1}{1+kx} \frac{\partial F}{\partial s} \right] = \sum \frac{(A_{m1} + mkA_{m-1,1})}{m!} x^m. \quad (27)$$

If one sets a particular solution

$$F = \sum \frac{C_m(s)}{m!} x^m \quad (28)$$

Then

$$\begin{aligned} C_{m+2} + (3m+1)kC_{m+1} + m(3m-1)k^2C_m + m(m-1)^2k^3C_{m-1} \\ - mk'C_{m-1} + C''_m + mkC''_{m-1} = \\ A_{m1} + 3mkA_{m-1,1} + 3m(m-1)k^2A_{m-2,1} + m(m-1)(m-2)k^3A_{m-3,1} \end{aligned} \quad (29)$$

This recursion relation gives for the first few terms

$$C_0 = C_1 = 0 \quad (30)$$

$$C_2 = A_{01} \quad (31)$$

$$C_3 = A_{11} - kA_{01} \quad (32)$$

$$C_4 = A_{21} - kA_{11} + 3k^2A_{01} - A''_{01} \quad (33)$$

CHOICE OF GAUGE

Since $\nabla^2 V = 0$, the vector potential may be written as

$$A = LU + \nabla(W - \frac{\partial V}{\partial y}) \quad (34)$$

or

$$A = LU + VT, \quad (35)$$

where, since W and $\partial V / \partial y$ are not needed separately they have been combined into T .

Since T may be any arbitrary function without altering the magnetic flux density, a gauge is selected whenever T is

specified. Having found U through the L-operator scheme, the simplest gauge is $T = 0$. This gives

$$A_x = \frac{1}{1+kx} \frac{\partial U}{\partial s} = - \frac{1}{1+kx} \left\{ \sum_{m,n} \frac{A'_{mn}}{m!(n+1)!} x^m y^{n+1} - \sum_m \frac{C'_m}{m!} x^m \right\} \quad (36)$$

$$A_y = 0$$

$$A_s = - \frac{\partial U}{\partial x} = \sum_m \frac{A_{m,n}}{(m-1)!(n+1)!} x^{m-1} y^{n+1} - \sum_m \frac{C_m}{(m-1)!} x^{m-1} \quad (37)$$

Another useful gauge is found by setting $xA_x + yA_y$ equal to zero. Let T be given by

$$T = \sum_m \frac{D_{mn}(s)}{m!n!} x^m y^n . \quad (38)$$

Then set $xA_x + yA_y = 0$ and obtain

$$\begin{aligned} & - \frac{x}{1+kx} \left\{ \sum_{m,n} \frac{A'_{mn}}{m!(n+1)!} x^m y^{n+1} - \sum_m \frac{C'_m}{m!} x^m \right\} \\ & + x \sum_m \frac{D_{mn}}{(m-1)!n!} x^{m-1} y^n + y \sum_m \frac{D_{mn}}{m!(n-1)!} x^m y^{n-1} = 0 \end{aligned}$$

or, after adjusting indices

$$\sum_{m,n} \frac{x^m y^n}{m!n!} \left\{ (m+n)D_{mn} + m(m+n-1)kD_{m-1,n-1} - mA'_{m-1,n-1} \right\} = - \sum_m \frac{mC'_{m-1}}{m!} x^m \quad (40)$$

Thus the recursion relation becomes for $n=0$:

$$D_{m0} + (m-1)D_{m-1,0} = -C'_{m-1} \quad (41)$$

or

$$D_{00} = D_{10} = D_{20} = 0 \quad (42)$$

$$D_{30} = -C'_2 \quad (43)$$

$$D_{40} = -C'_3 + 3kC'_2 \quad (44)$$

$$D_{50} = -C'_4 + 4kC'_3 - 12k^2C'_2 \quad (45)$$

For $n = 1, 2, 3, \dots$, etc. the recursion relation becomes

$$(m+n)D_{mn} + m(m+n-1)kD_{m-1,n} = mA'_{m-1,n-1} \quad (46)$$

or for $n = 1$

$$D_{01} = 0 \quad (47)$$

$$D_{11} = \frac{1}{2} A'_{00} \quad (48)$$

$$D_{21} = \frac{2}{3} A'_{10} - \frac{2}{3} k A'_{00} \quad (49)$$

$$D_{31} = \frac{3}{4} A'_{20} - \frac{3k}{2} A'_{10} + \frac{3k^2}{2} A'_{00} \quad (50)$$

$$D_{41} = \frac{4}{5} A'_{30} - \frac{12}{5} k A'_{20} + \frac{24}{5} k^2 A'_{10} - \frac{24}{5} k^3 A'_{00} \quad (51)$$

for $n = 2$

$$D_{02} = 0 \quad (52)$$

$$D_{12} = \frac{1}{3} A'_{01} \quad (53)$$

$$D_{22} = \frac{1}{2} A'_{11} - \frac{1}{2} k A'_{01} \quad (54)$$

$$D_{32} = \frac{3}{5} A'_{21} - \frac{6}{5} k A'_{11} + \frac{6}{5} k^2 A'_{01} \quad (55)$$

$$D_{42} = \frac{2}{3} A'_{31} - 2k A'_{21} + 4k^2 A'_{11} - 4k^3 A'_{01} \quad (56)$$

and, for $n = 3$

$$D_{03} = 0 \quad (57)$$

$$D_{13} = \frac{1}{4} A'_{02} \quad (58)$$

$$D_{23} = \frac{2}{5} A'_{12} - \frac{2}{5} k A'_{02} \quad (59)$$

$$D_{33} = \frac{1}{2} A'_{22} - k A'_{12} + k^2 A'_{02} \quad (60)$$

$$D_{43} = \frac{4}{7} A'_{32} - \frac{12}{7} k A'_{22} + \frac{24}{7} k^2 A'_{12} - \frac{24}{7} k^3 A'_{02} \quad (61)$$

Then, in this gauge

$$A_x = \frac{1}{1+kx} \frac{\partial U}{\partial s} + \frac{\partial T}{\partial x} \quad (62)$$

$$A_y = \frac{\partial T}{\partial y} \quad (63)$$

$$A_s = - \frac{\partial U}{\partial x} + \frac{1}{1+kx} \frac{\partial T}{\partial s} \quad (64)$$

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